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On an assertion of J. Rhodes and the finite basis and finite vertex rank problems for pseudovarieties

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Abstract

J. Rhodes asserted at in Braga in 1997, in response to a question of J. Almeida, that $\mathbf{A} * \mathbf{G}$ is not finite vertex rank. We prove his assertion and more. By way of contrast, we show that $\mathbf{G} * \mathbf{A}$ is local, i.e. has vertex rank 1.

MSC: 20M07

1. Introduction

Tilson's derived category theorem [27] established a strong connection between computing semidirect product decompositions of monoids and computing category membership in pseudovarieties of monoids (also called computing the global of a pseudovariety). Since then the computation of globals of pseudovarieties has become an important aspect of finite semigroup theory and we could easily provide a long list of papers on the subject (see [27,5] for some references). In particular, Knast's famous result on dot-depth one [12] can be interpreted as a computation of the global of $\mathbf{J}$.

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Of particular interest is the locality of a pseudovariety. A pseudovariety of monoids is said to be local [27] if category membership in **V** reduces to checking membership in **V** for the local monoids of the category. Again we could provide a list of articles dealing with the locality of various pseudovarieties; see, for instance, the bibliography of [5,10,27]. Similar notions exist for pseudovarieties of semigroups, only the role of categories is instead played by semigroupoids.

Rhodes showed in 1976 [16] that the pseudovariety of complexity one semigroups (see [8,14,15] for the definition) is not local. He also showed that the complexity one-half pseudovariety $\mathbf{A} * \mathbf{G} = \mathbf{A} \textcircled{m} \mathbf{G}$ is not local, although this latter result can easily be deduced from earlier examples of Rhodes and Tilson [18]. Here **A** is the pseudovariety of aperiodic semigroups (monoids) and **G** is the pseudovariety of all finite groups. The reader is referred to [8,10] for the definitions of the semidirect product ($*$) and the Malcev product ($\textcircled{m}$). Unpublished work of Rhodes shows that no (integral) complexity pseudovariety is local.

Almeida's work on hyperdecidability [1] and its improvement tameness [3,4] allow one to deduce decidability results for semidirect products of the form $\mathbf{V} * \mathbf{W}$ if **V** has finite vertex rank, **gV** (the global of **V**) is decidable and **W** is hyperdecidable or tame. A pseudovariety **U** of categories (semigroupoids) is said to have vertex rank n if a category (semigroupoid) C belongs to **U** if and only if all its subcategories (subsemigroupoids) with at most n vertices belong to **U**. A pseudovariety has finite vertex rank if it has vertex rank n for some integer n. It is easy to see that the property of having finite vertex rank is equivalent to that of being definable by pseudoidentities over graphs having at most n vertices [1,5]. In particular, any finitely based pseudovariety of categories has finite vertex rank. The converse is not true. For instance, **EA** is local [20], but **EA** and hence **gEA** is not finitely based [28] (a semigroup belongs to **EA** if and only if its idempotents generate an aperiodic semigroup).

All definitions and terminology involving vertex rank can be transferred to pseudovarieties of monoids and semigroups via the correspondence $\mathbf{V} \leftrightarrow \mathbf{gV}$ in the obvious way. So, for instance, **V** is local if and only if it has vertex rank 1. We use the term infinite vertex rank as a synonym for not finite vertex rank.

Besides the results involving hyperdecidability and tameness mentioned earlier, the importance of finite vertex rank also lies in the fact that the Almeida-Weil basis theorem for semidirect products [5] only has a valid proof for $\mathbf{V} * \mathbf{W}$ with **V** finite vertex rank; see [17,29].

Almeida and Azevedo constructed in [2] some pseudovarieties of commutative semigroups which do not have finite vertex rank.

We prove here an assertion made by J. Rhodes that $\mathbf{A} * \mathbf{G}$ does not have finite vertex rank. Hence the techniques of hyperdecidability [1] and tameness [3,4] do not seem to apply to complexity [13–15,25].

In fact using what is essentially a primitive form from [18] of the presentation lemma [7] (see also [22]), we show that any pseudovariety **V** in the interval $[\mathbf{A} \vee \mathbf{G}, \mathbf{A} * \mathbf{G}]$ has infinite vertex rank. More generally we show that if **H** is a pseudovariety of groups, $G \notin \mathbf{H}$ is a group, and $\overline{\mathbf{H}}$ is the pseudovariety of monoids (semigroups) whose subgroups belong to **H**, then every pseudovariety in the interval $[\mathbf{A} \vee \langle G \rangle, \overline{\mathbf{H}} * \mathbf{G}]$ is not finite vertex rank (recall: $\overline{\mathbf{H}} * \mathbf{G} = \overline{\mathbf{H}} \textcircled{m} \mathbf{G}$ since $\overline{\mathbf{H}}$ is local (c.f. [20])). In particular

$\overline{\mathbf{H}}*\mathbf{G}$ is not local. In fact $\mathbf{A}$ can be replaced by a certain locally finite subpseudovariety, as we shall see below.

We take advantage of the coordinate approach to Type II elements used by Rhodes and Tilson in [8] and Graham normalizations [9] of Rees matrix semigroups. This will allows us to almost entirely avoid computations by instead drawing pictures. While we shall summarize the results we use, the reader is encouraged to read [18] or to consult Tilson's excellent survey article [24] before reading this paper. This is a conscious choice. One could get away with a direct argument using that the Type II subsemigroup of a finite semigroup is its self-conjugate core. This approach, while more elementary, requires more computations and obscures the intuition behind the construction.

In the final section, we show, by way of contrast, that $\mathbf{G}*\mathbf{A}$ (and more generally $\mathbf{G}*\overline{\mathbf{H}}$ where $\mathbf{H}$ a pseudovariety of groups closed under extension) is local.

The paper is organized as follows. First we prove our main results assuming the construction of certain monoids; then we proceed to construct these monoids.

2. Finite bases and infinite vertex rank

In this paper, all monoids, semigroups, categories, and semigroupoids are assumed finite. We begin with the following proposition.

Proposition 2.1. *Let* $\mathbf{H}$ *be a pseudovariety of groups and let* $G \notin \mathbf{H}$ *be a group. Then for each* $N > 0$, *there exists a category* K_N *such that*:

1. *Every subcategory of* K_N *with at most* N *vertices belongs to* $\mathbf{g}(\mathbf{A} \vee \langle G \rangle)$;
2. $K_N \notin \mathbf{g}(\overline{\mathbf{H}} * \mathbf{G})$.

Proof. Postponed. □

Theorem 2.2. *Let* $\mathbf{H}$ *be a pseudovariety of groups and* $G \in \mathbf{G} \setminus \mathbf{H}$. *Then each pseudovariety in the interval* $[\mathbf{A} \vee \langle G \rangle, \overline{\mathbf{H}} * \mathbf{G}]$ *has infinite vertex rank.*

Proof. Suppose $\mathbf{V}$ in this interval has vertex rank N and consider K_N from Proposition 2.1. All N vertex subcategories of K_N belong to $\mathbf{V}$, but K_N does not, contradicting that $\mathbf{V}$ has vertex rank N. □

In the above result, we may consider pseudovarieties of categories and semigroupoids instead. We have the following corollary.

Corollary 2.3. *If* $\mathbf{H}$ *is a non-trivial pseudovariety of groups*, $\mathbf{A} \vee \mathbf{H}$ *has infinite vertex rank. If* $\mathbf{H}$ *is a proper pseudovariety of groups*, $\overline{\mathbf{H}} * \mathbf{G}$ *has infinite vertex rank. In particular*, $\mathbf{A} \vee \mathbf{G}$ *and* $\mathbf{A} * \mathbf{G}$ *have infinite vertex rank.*

Since $\overline{\mathbf{H}}$ and $\mathbf{G}$ are local (the latter only as a pseudovariety of categories), it follows that not only does the semidirect product operator fail to preserve locality, but that in fact the semidirect product of local pseudovarieties can be of infinite vertex rank.

To prove Theorem 2.2 and our results on finite bases, we need the following definition and proposition. Let M be a monoid. Say that $m \in M$ is *indecomposable* if $m = ab$ implies $a = 1$ or $b = 1$; we use $Ind(M)$ to denote the set of indecomposable elements of M. If $m \in Ind(M)$, then $M \setminus m$ is a submonoid.

Proposition 2.4. *Let* $\mathbf{H}$ *be a pseudovariety of groups and* $G \in \mathbf{G} \setminus \mathbf{H}$. *Then, for each odd integer* $n > 1$, *there exists a monoid* M_n *with n indecomposable elements* $t_1, \ldots, t_n$ *satisfying*:

1. $M_n \in \mathbf{EA}$;
2. $M_n \setminus t_j \in \mathbf{A} \vee \mathbf{G}$, $j = 1, \ldots, n$;
3. $M_n \notin \overline{\mathbf{H}} * \mathbf{G}$.

Proof. Postponed. □

Recall that $\mathbf{EV}$ is the pseudovariety of monoids whose idempotents generate a submonoid in $\mathbf{V}$.

In fact, the construction of the M_n is such that $\mathbf{A}$ can be replaced in all our results by its locally finite subpseudovariety $\mathbf{ACS}^0 \, \textcircled{m} \, \mathbf{N}_2$ where $\mathbf{N}_2$ is the pseudovariety generated by null semigroups with adjoined identities and $\mathbf{ACS}^0$ is the pseudovariety generated by aperiodic 0-simple semigroups.

We now recover a result of Volkov [28].

Theorem 2.5. *Let* $\mathbf{H}$ *be a pseudovariety of groups and* $G \in \mathbf{G} \setminus \mathbf{H}$. *Then no pseudovariety* $\mathbf{V} \in [\mathbf{A} \vee \langle G \rangle, \overline{\mathbf{H}} * \mathbf{G}]$ *is finitely based.*

Proof. Suppose $\mathbf{V}$ has a basis of pseudoidentities in k variables and $n > k$ is an odd number. Let T be a k-generated submonoid (subsemigroup) of M_n (say with generating set X). Then there must be $1 \leqslant j \leqslant n$ such that $t_j \notin X$. It follows that $T \leqslant M_n \setminus t_j \in \mathbf{A} \vee \langle G \rangle$ by Proposition 2.4. Thus all k-generated submonoids (subsemigroups) of M_n belong to $\mathbf{V}$. But M_n does not by Proposition 2.4, a contradiction. □

In fact, our results are even stronger than Volkov's for this interval. He shows that the whole interval $[\mathbf{A} \vee \mathbf{G}, \mathbf{EA}]$ is not finitely based. However, since our monoids M_n belong to $\mathbf{EA}$, it follows that no pseudovariety $\mathbf{V}$ in the interval $[\mathbf{A} \vee \mathbf{G}, \mathbf{A} * \mathbf{G}]$ is finitely based as a subpseudovariety of $\mathbf{EA}$ (meaning that a basis for $\mathbf{V}$ cannot be obtained by adding finitely many pseudoidentities to a basis for $\mathbf{EA}$). Similar comments apply replacing $\mathbf{A}$ with $\overline{\mathbf{H}}$ for $\mathbf{H}$ a proper pseudovariety of groups.

2.1. Proof of Proposition 2.1 (using Proposition 2.4)

Choose an odd integer $n > \binom{N}{2}$ and consider the monoid M_n guaranteed by Proposition 2.4.

Let $\{m_1, \ldots, m_k\}$ be a generating set for M_n. Since $t_1, \ldots, t_n \in Ind(M_n)$, it follows that each t_i is included in this generating set. Without loss of generality, we may assume

$m_i = t_i$ for $1 \leqslant i \leqslant n$. Let $\{e_1, \ldots, e_k\}$ be a basis for the vector space $\mathbb{Z}_2^k$ and define $\varphi : M_n \to \mathbb{Z}_2^k$ to be the relational morphism generated by the function

$$m_i \mapsto e_i.$$

Let K_N be the category $\mathbf{Der}(\varphi)$ defined in [23] (this is essentially the unfactored derived category of [27]). First we claim $\mathbf{Der}(\varphi) \notin \mathbf{g}(\overline{\mathbf{H}} * \mathbf{G})$.

Suppose $\mathbf{Der}(\varphi) \in \mathbf{g}(\overline{\mathbf{H}} * \mathbf{G})$. Then, since the derived category of φ [27,23] is a quotient of $\mathbf{Der}(\varphi)$, the Derived Category Theorem [27] tells us

$$M_n \in (\overline{\mathbf{H}} * \mathbf{G}) * \mathbf{G} = \overline{\mathbf{H}} * \mathbf{G},$$

contradicting Proposition 2.4.

Now consider the faithful morphism $\theta : \mathbf{Der}(\varphi_n) \to M_n$ given by

$$(g_L, (m, g)) \mapsto m.$$

Since each t_j is indecomposable,

$$t_j\theta^{-1} = \{(g, (t_j, e_j)) \mid g \in \mathbb{Z}_2^k\}. \tag{1}$$

Let C be a subcategory with at most N vertices. Abusing notation, let $\theta : C \to M_n$ be the restriction (which is also faithful). We show that there exists $1 \leqslant j \leqslant n$ so that $C\theta \subseteq M_n \setminus t_j$. It will then follow that C divides $M_n \setminus t_j$ and hence, by Proposition 2.4, $C \in \mathbf{g}(\mathbf{A} \vee \langle G \rangle)$, as desired.

By (1), we just need to show that there exists $1 \leqslant j \leqslant n$ such that no edge $(g, (t_j, e_j))$ belongs to C. Notice that $(g, (t_j, e_j))$ goes from g to $g + e_j$. Thus the index j is uniquely determined by the endpoints of the edge. In particular, the number of indices j such that an edge $(g, (t_j, e_j))$ can appear in C is at most $\binom{N}{2}$. Since $n > \binom{N}{2}$, it follows that $t_j \notin C\theta$ for some j and so $C\theta \subseteq M_n \setminus t_j$, as desired. $\square$

3. Type II, Graham normalizations and other preliminaries

We now aim to construct the desired semigroups M_n. First we need some preliminary notions. The reader is referred to [15] for basic semigroup theory and notions therein shall be used without hesitation. We follow Eilenberg [8] with respect to wreath products of partial transformation semigroups.

Recall that an element of a semigroup S is said to be of *Type* II [18,10] if it relates to 1 under all relational morphisms with a group. The collection of Type II elements is a subsemigroup S_{II} of S. This subsemigroup was shown by Ash [6] (and independently by Ribes and Zalesskiĭ [19]) to be the smallest subsemigroup containing the idempotents and closed under weak conjugation (the self-conjugate core); see [10] for the relevant definitions and for a survey of related results. The case of regular elements was first handled by Rhodes and Tilson [18,26]. We shall never need more than the results of [18] in this paper. The old Rhodes–Tilson approach is a Rees coordinate version of the geometric approaches taken in later papers [26,21].

We now describe how to compute the Type II elements of a regular $\mathscr{J}$-class J of a semigroup S in Rees coordinates. This procedure is a summary of the results in [18]; see also the survey [24].

Let A be the set of $\mathscr{R}$-classes of J and B the set of $\mathscr{L}$-classes; let G be a maximal subgroup. Two $\mathscr{L}$ classes b_1, b_2 are said to be *attached* if there is an $\mathscr{R}$-class a such that $b_1 \cap a$ and $b_2 \cap a$ are groups. If we choose a Rees matrix representation $\mathscr{M}^0(G,A,B,C)$ for J^0, then b_1, b_2 are attached if and only if there is an element $a \in A$ such that $b_1Ca \neq 0 \neq b_2Ca$ (notice that we write C in the middle of its two arguments to reflect the symmetry of the situation). Being attached is a reflexive and symmetric relation. We use *TCA* to denote the transitive closure of being attached (TA is used in [18]). By a *TCA block*, we mean a block of the TCA partition. Similarly, attached and TCA can be defined for elements of A. The TCA blocks of A and B are linked in the following way. For each TCA block B_i, there is a unique TCA block A_i such that $bCa \neq 0$ for some $b \in B_i$ and $a \in A_i$. We shall say that B_i and A_i are in this case *attached*. Sometimes the $A_i \times B_i$ will be referred to as the *TCA blocks* of C. All multi-defined terminology should be clear from the context.

The easiest way to think about this is to construct a bipartite graph [9] with vertices $A \cup B$ and with an edge from a to b if $bCa \neq 0$. The connected components of this graph correspond exactly to the TCA blocks of C.

If we arrange the TCA blocks in order $A_1 \times B_1, \ldots, A_n \times B_n$ and permute the rows and columns of C accordingly, we obtain a new coordinatization of J^0 for which the structure matrix is block diagonal and each block is a regular Rees matrix. Graham proved further the following result [9].

Theorem 3.1 (Graham). *There exists an effectively constructible Rees matrix representation $\mathscr{M}^0(G,A,B,C)$ of J^0 such that*:

1. *C is the direct sum of regular $B_i \times A_i$ matrices C_i (where the $A_i \times B_i$ are the attached TCA-blocks)*

$$C = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & C_n \end{pmatrix}$$

2. $\langle E(J^0)\rangle = \bigcup \mathscr{M}^0(G_i, A_i, B_i, C_i)$, *where G_i is the subgroup of G generated by the entries of C_i.*

In particular, $\langle E(J^0)\rangle \in \mathbf{A}$ if and only if J^0 has a Rees matrix representation in which all entries are either 0 *or* 1.

Such a Rees matrix representation is called a *Graham normalization*. In general we use the word *normalization* to mean a Rees coordinatization of a regular $\mathscr{J}$-class.

We recall [15] that all renormalizations of a regular $\mathscr{J}$-class that do not permute rows or columns are of the form

$$(a,g,b) \mapsto (a, g_a g g_b, b) \tag{2}$$

where $a \mapsto g_a, b \mapsto g_b$ are functions $A \to G$, $B \to G$, respectively. If $C : B \times A \to G^0$ is the original matrix, then the matrix $C' : B \times A \to G^0$ in the new coordinate system is given by

$$bC'a = g_b^{-1}(bCa)g_a^{-1}. \tag{3}$$

Given $s \in S$ and a partition P of B, we can define a relation $R_s : B/P \to B/P$ as follows. If B_i is a partition block, then $B_i R_s B_j$ if and only if $B_i s \cap B_j \neq \emptyset$. Here $B_i s$ makes sense since B_i is a union of $\mathscr{L}$-classes and the $\mathscr{L}$-relation is right compatible and so S acts on the $\mathscr{L}$-classes of J in a natural way [15]. We say that the partition P of B is *weakly preserved* if R_s is a partial injective function for all S. For instance, $P = B$ is a *weakly preserved partition*. It is easy to see that any elements which are in the same TCA block must be in the same block of any weakly preserved partition; that is the blocks of a weakly preserved partition are unions of TCA blocks. The intersection of weakly preserved partitions is again weakly preserved [18] and so the weakly preserved partitions form a lattice (the order is $P \leqslant P'$ if $B/P \to B/P'$ is well defined). The unique smallest weakly preserved partition is called the *Type* II *partition*; see [18]. We denote it by P_{II}. The blocks of the Type II partition are called *Type* II *blocks*.

Dual notions exist for partitions of A. Moreover, given a weakly preserved partition P of B, we can obtain a partition P' of A by saying that if block i of P is a union of a certain collection of TCA blocks of B, then block i of P' is the union of the corresponding TCA blocks of A. The linked equations [15] then imply that P' is weakly preserved, as well; see [18]. The *Type* II *partition* of A corresponds in this manner to the Type II partition of B. In [18] they work with both partitions simultaneously. Sometimes if A_i and B_i are corresponding Type II blocks, then $A_i \times B_i$ is also called a *Type* II *block*.

If $A_1 \times B_1, \ldots, A_t \times B_t$ are the blocks of the Type II partition, then a Graham normalization $\mathscr{M}^0(G, A, B, C)$ is said to be a *Type* II *normalization* if

$$\bigcup \mathscr{M}^0(N_0, A_i, B_i, C_i) \setminus 0 \subseteq S_{\mathrm{II}} \cap J$$

where the $A_i \times B_i$ are the Type II blocks, C_i is the regular matrix obtained by restricting C to $B_i \times A_i$ and N_0 is the normal subgroup generated by the non-zero matrix entries of C (recall that we are dealing with a Graham normalization) [18]. If $X \subseteq G$, we use $\langle X \rangle^G$ to denote the normal subgroup of G generated by X.

Not all Graham normalizations are Type II normalizations [18] (and this fact will be exploited in the sequel). However, after finding a Type II normalization, it is quite easy to describe the Type II elements of J.

Recall that if $s \in S$, then the action of s on the right of J can be represented by a $B \times B$-row monomial matrix $RM_J(s)$ over G^0; this is none other than the classical right Schützenberger representation of S on J [15]. The associated transformation semigroup is denoted $(J, RM_J(S))$ (RM is for *right mapping* [15]).

For $s \in S$, $RLM_J(s)$ denotes the matrix obtained from $RM_J(s)$ by setting all non-zero entries equal to 1. In this manner one obtains a transformation semigroup $(B, RLM_J(S))$ (RLM is for *right letter mapping* [15]).

The matrix $RM_J(s)$ can be encoded by the pair $(f_s, RLM_J(s))$ where

$$f_s : B \to G$$

is given by setting bf_s equal to the unique non-zero element of G in row b of $RM_J(s)$ if such an element exists, and to an arbitrary element of G, otherwise. In this way, we have represented $RM_J(s)$ as an element of the wreath product [15,8] $G \wr (B, RLM_J(S))$ where

$$(a,g,b)s \text{ is defined if and only if } bs \text{ is defined, and in this case}$$

$$(a,g,b)s = (a, g(bf_s), bs).$$

(Here, and from now on, we drop the notation $RLM_J(s)$ and simply write s or $\cdot s$). In conclusion, we see $RM_J(S) \leqslant G \wr (B, RLM_J(S))$.

If $\mathscr{M}^0(G,A,B,C)$ is a Type II normalization, then one has [18]:

Theorem 3.2 (Rhodes/Tilson).

$$S_{\mathrm{II}} \cap J = \bigcup \mathscr{M}^0(N_{\mathrm{II}}, A_i, B_i, C_i) \setminus 0,$$

where

$$N_{\mathrm{II}} = \langle (bf_s)^{-1}(\bar{b}f_s) \mid b, \bar{b} \in B_i \in P_{\mathrm{II}}, bs, \bar{b}s \in B \rangle^G.$$

Equivalently, N_{II} *is the normal closure of* N_0 *and all elements of the above form where* $s >_{\mathscr{J}} J$.

In [18] an algorithm is given to construct iteratively a Type II normalization; the reader is also referred to [24]. The procedure is as follows. One starts with the TCA partition and an arbitrary Graham normalization. If the TCA partition is weakly preserved, the algorithm stops because we already have a Type II normalization. Otherwise choose s (necessarily $>_{\mathscr{J}} J$) such that there are $b, \bar{b}$ in the same block with $bs, \bar{b}s$ in different blocks or dually $b, \bar{b}$ in the different blocks and $bs, \bar{b}s$ in the same block. One takes the union of the blocks of $bs, \bar{b}s$ in the former case and of the blocks of $b, \bar{b}$ in the latter. If bf_s and $\bar{b}f_s$ are different modulo N_0, then a certain renormalization is performed. Since in our case we never have to perform such a renormalization, we merely refer the reader to [18] for details.

One then iterates the procedure until no such s exists. At this point one has obtained the Type II partition with a Type II normalization [18].

Recall that a semigroup S is called *right mapping* [15] if it has a (unique) 0-minimal I on which it acts faithfully on the right. Stronger versions of the following result can be found in [7,18,22].

Proposition 3.3. *Let* S *be a right mapping semigroup with distinguished* 0-*minimal ideal* $I = \mathscr{M}^0(G,A,B,C)$. *Suppose furthermore that, for all* $s \in S \setminus 0$, *there exists* $g_s \in G$ *such that* $Bf_s \subseteq \{g_s\}$. *Then*

$$S \leqslant G \times (B, RLM_J(S)),$$

where $J = I \setminus 0$. *In particular, if* $RLM_J(S) \in \mathbf{A}$, *then* $S \in \mathbf{A} \vee \langle G \rangle$.

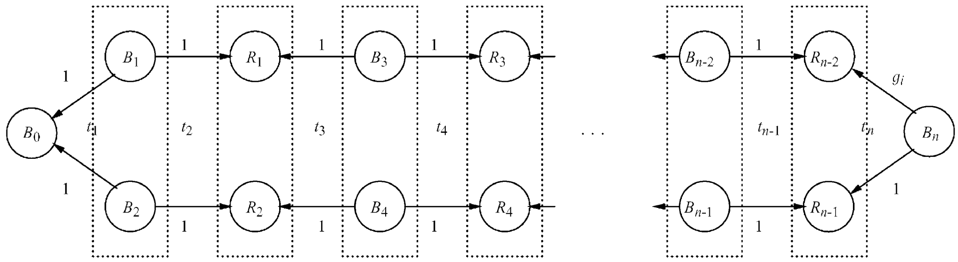

Fig. 1. TCA block graph for B_i.

Proof. The hypotheses simply state that the usual embedding of S into $G \wr (B, RLM_J(S))$ actually puts S in $G \times (B, RLM_J(S))$. Indeed, our assumptions simply translate to stating that, for each $s \in S$, the row monomial matrix $RM_J(s)$ is a scalar multiple of $RLM_J(s)$ by g_s. $\square$

In order to apply the above proposition, it is convenient to use the following straightforward lemma [15,18,7].

Lemma 3.4. *Let S be a right mapping semigroup with distinguished ideal $I = \mathscr{M}^0(G, A, B, C)$ and let $s = (a_0, g_0, b_0) \in I$. Then $bf_s = (bCa_0)g_0$.*

4. The construction of the M_n

Recall that a non-trivial semigroup is called *group mapping* [15] if it acts faithfully on the left and right of a (necessarily unique) 0-minimal ideal I containing a non-trivial group.

Fix a non-trivial group G and choose $g_1, \ldots, g_m \in G$ such that

$$\langle g_1, \ldots, g_m \rangle^G = G.$$

Fix an odd number $n > 1$. We construct a group mapping monoid M_n with aperiodic right letter mapping semigroup (with respect to its distinguished $\mathscr{J}$-class), trivial group of units, and distinguished 0-minimal ideal I as follows.

We begin with an intuitive sketch of the proof. The key idea is to associate to each g_i a collection of TCA blocks

$$B_0^i, B_1^i, R_1^i, \ldots, B_{n-1}^i, R_{n-1}^i, B_n^i$$

whose purpose is to put (in conjunction with some indecomposable null elements $t_1, \ldots, t_n$) g_i into $(M_n)_{\mathrm{II}}$. Recall that an element is called *null* if it is not regular [15].

The t_j act on these TCA blocks as indicated in Fig. 1 (which the reader is advised to refer to frequently). The dashed lines represent the Type II partition of B_i (the union of the TCA blocks considered above). The labels on the arrows show how the group coordinate is effected by the t_j. *In this figure and all subsequent figures, the* *superscript i is omitted*. We shall call the blocks B^i_j, R^i_j with $1 \leqslant j \leqslant n-2$ odd the *top* row and with $2 \leqslant j \leqslant n-1$ even the *bottom* row.

The elements $t_1, \ldots, t_{n-1}$ serve the purpose of creating the Type II partition on this collection of $\mathscr{L}$-classes *without* any renormalization required. Then t_n puts g_i into the Type II subsemigroup since it acts as the identity on one element of B^i_n and g_i on another. However, if t_j is removed for some $j < n$, then we shall be able to perform a renormalization to which the hypotheses of Proposition 3.3 apply. Hence the monoid $M_n \setminus t_j$ will be in $\mathbf{A} \vee \langle G \rangle$ for all j.

More formally, associated to each g_i, we have sets:

$$A^i_0 = \{x^i, y^i\}$$

$$\text{and, for } 1 \leqslant j \leqslant n, \quad A^i_j = \{u^i_j, v^i_j, w^i_j\}$$

$$\text{and, for } 1 \leqslant j \leqslant n-1, \quad r^i_j = \{r^i_j\}$$

$$B^i_0 = \{X^i, Y^i, Z^i\}$$

$$\text{and, for } 1 \leqslant j \leqslant n, \quad B^i_j = \{U^i_j, V^i_j\}$$

$$\text{and, for } 1 \leqslant j \leqslant n-1, \quad R^i_j = \{R^i_j\}.$$

Notice that we are abusing notation by using the same letter to denote a singleton TCA-block and its unique member. In this notation, we use uppercase letters for $\mathscr{L}$-classes and lowercase letters for $\mathscr{R}$-classes. The subscript indicates the TCA-block.

To decongest notation, we shall suppress the superscript i whenever there is no danger of confusion. We define $C^i_0 : B^i_0 \times A^i_0 \to G^0$ by

$$C^i_0 = \begin{array}{c|cc} & x & y \\ \hline X & 1 & 0 \\ Y & 0 & 1 \\ Z & 1 & 1 \end{array} \tag{4}$$

$C^i_j : B^i_j \times A^i_j \to G^0$, $1 \leqslant j \leqslant n$, by

$$C^i_j = \begin{array}{c|ccc} & u_j & v_j & w_j \\ \hline U_j & 1 & 0 & 1 \\ V_j & 0 & 1 & 1 \end{array} \tag{5}$$

and $D^i_j : R^i_j \times r^i_j \to G^0$, $1 \leqslant j \leqslant n-1$, by

$$D^i_j = \begin{array}{c|c} & r_j \\ \hline R_j & 1 \end{array} \tag{6}$$

Let

$$A_i = \bigcup (A^i_0 \cup A^i_j \cup r^i_j) \quad \text{and} \quad B_i = \bigcup (B^i_0 \cup B^i_j \cup R^i_j). \tag{7}$$

The matrix $C_i : B_i \times A_i \to G^0$ is then the direct sum of the matrices C^i_0, C^i_j, D^i_j defined in (4)–(6). Notice that the TCA-blocks of C_i are the various sets considered above. Also notice that the matrix entry corresponding to an uppercase letter and its corresponding lowercase letter with the same subscript (and superscript) is always 1.

Let $A = \bigcup A_i$ and $B = \bigcup B_i$. The matrix $C : B \times A \to G^0$ is then the direct sum of the C_i. The regular Rees matrix semigroup $\mathscr{M}^0(G, A, B, C)$ will be denoted by I. The TCA blocks of C consist of the TCA blocks of the various C_i. Notice that $\langle E(I) \rangle$ is aperiodic since C consists of just 0's and 1's.

Our null elements $t_1, \ldots, t_n$ are defined as follows. The product of any two (not necessarily distinct) elements t_j, t_k is defined to be zero. We define the action of t_j on the right and left of $J = I \setminus 0$ in coordinates (c.f. [15]). All undefined products are taken to be zero. Let us fix an i so we may drop all superscripts.

We begin with t_1:

$$(a, g, U_1)t_1 = (a, g, X), \quad (a, g, U_2)t_1 = (a, g, Y),$$
$$t_1(x, g, b) = (u_1, g, b), \quad t_1(y, g, b) = (u_2, g, b). \tag{8}$$

For t_n, we have:

$$(a, g, U_n)t_n = (a, gg_i, R_{n-2}), \quad (a, g, V_n)t_n = (a, g, R_{n-1}),$$
$$t_n(r_{n-2}, g, b) = (u_n, g_i g, b), \quad t_n(r_{n-1}, g, b) = (v_n, g, b). \tag{9}$$

For t_j with $1 < j < n$, we must divide into the cases of j odd and j even. The case $k = 0$ corresponds to the top row and $k = 1$ to the bottom row.

For j odd:

$$(a, g, U_{j+k})t_j = (a, g, R_{j-2+k}), \quad k = 0, 1,$$
$$t_j(r_{j-2+k}, g, b) = (u_{j+k}, g, b), \quad k = 0, 1. \tag{10}$$

For j even:

$$(a, g, V_{j-k})t_j = (a, g, R_{j-k}), \quad k = 0, 1$$
$$t_j(r_{j-k}, g, b) = (v_{j-k}, g, b), \quad k = 0, 1. \tag{11}$$

Observe that the only non-trivial group coordinate action is given (in wreath product coordinates) by $U_n f_{t_n} = g_i$.

The associativity of M_n is straightforward. The only cases to check are things of the form $(a, g, b)t_j(a', g', b')$; this amounts to verifying what are called the linked equations in [15]. For instance,

$$[(a, g, U_n)t_n](r_{n-2}, g', b) = (a, gg_i, R_{n-2})(r_{n-2}, g', b) = (a, gg_i g', b),$$

$$(a, g, U_n)[t_n(r_{n-2}, g', b)] = (a, g, U_n)(u_n, g_i g', b) = (a, gg_i g', b).$$

The other verifications are similar.

Observe that $RLM_J(M_n)$ [15] is aperiodic. Indeed, $RLM_J(M_n)$ is obtained from M_n by replacing G with $\{1\}$ and adjusting the actions of the t_n accordingly, since C consists of 0's and 1's, M_n is group mapping and no two t_j act the same on B (see Lemma 3.4 and Fig. 1). In fact, $\mathscr{H}$ is a congruence on M_n and $RLM_J(M_n) = M_n/\mathscr{H}$.

5. Proof of Proposition 2.4

First note that $M_n \in \mathbf{EA}$ since C contains only 0 and 1 as entries and 1 is the only regular element outside of I. Clearly $t_1, \dots, t_n$ are indecomposable.

To show $G \leqslant (M_n)_{\mathrm{II}}$, we use the approach of [18]. A direct argument involving the self-conjugate core could be given, but we prefer our picture proof. However later, we shall sketch such an argument. [2]

Proposition 5.1. *The Type* II *partition of* B *is*

$$\{B_0^i, \{B_j^i \cup B_{j+1}^i\}, \{R_j^i \cup R_{j+1}^i\}, B_n^i \mid 1 \leqslant j \leqslant n-2 \ odd,\ 1 \leqslant i \leqslant m\}.$$

The Type II *partition of* A *is described analogously.*

Proof. This is clear from Fig. 1. □

Also note that we can obtain the Type II partition of B using just the t_j with $j < n$, all i; see Fig. 1. Moreover, no renormalizations are needed in performing the algorithm from [18] to obtain the Type II normalization since the t_j with $j < n$ act as the identity on group coordinates (when defined).

Notice that (suppressing superscripts)

$$U_n t_n, V_n t_n \in B$$

by (9) and that U_n, V_n belong to the same Type II partition block (in fact the same TCA block). Also

$$(V_n f_{t_n})^{-1} U_n f_{t_n} = g_i. \tag{12}$$

Hence by Theorem 3.2 and (12), $(M_n)_{\mathrm{II}} \cap J$ consists of all elements (a, g, b) with a, b in corresponding partition blocks of the Type II partition and $g \in \langle g_1, \dots, g_m \rangle^G = G$. Thus $G \subseteq (M_n)_{\mathrm{II}}$. In particular, if $G \notin \mathbf{H}$, then $(M_n)_{\mathrm{II}} \notin \overline{\mathbf{H}}$.

Using that $S \in \overline{\mathbf{H}} ⓜ \mathbf{G}$ if and only if $S_{\mathrm{II}} \in \overline{\mathbf{H}}$ and recalling that $\overline{\mathbf{H}} ⓜ \mathbf{G} = \overline{\mathbf{H}} * \mathbf{G}$ (by the locality of $\overline{\mathbf{H}}$ [10,20]), we see that we have proven the following.

Proposition 5.2. $M_n \notin \overline{\mathbf{H}} * \mathbf{G}$.

It is natural to ask how to represent G as a subgroup of the self-conjugate core of M_n. We focus on a single g_i since obvious weak conjugations then allow us to put G in a single maximal subgroup of M_n (see [18]). Let us then omit the subscripts and superscripts i. We sketch how to show that $s = (u_n, g, U_n)$ is in the self-conjugate core of M_n.

The methods of [26,21] state that it suffices to find a Dyck word labeling a path in the Schützenberger graph of the $\mathscr{R}$-class u_n from the idempotent $e = (u_n, 1, U_n)$ to s. One then replaces inverses of letters in the labeling by appropriately chosen weak inverses.

[2] We thank K. Auinger for suggesting this and showing us the calculations for $n = 7$.

Fig. 1 is the quotient of the Schüztenberger graph of the $\mathscr{R}$-class u_n by G. The circuit at B_n labeled by the Dyck word

$$t_n t_{n-1}^{-1} t_{n-2} t_{n-3}^{-1} \cdots t_1 t_1^{-1} t_2 \cdots t_{n-1} t_n^{-1}$$

almost gives us what we want; however, to move within each TCA block (i.e. vertex of Fig. 1), we must multiply by certain elements of $\langle E(I) \rangle$ [18]. Since each block R_j is a singleton, there is no need to move about in the R_j. Thus one can show that

$$s = e_n t_n t'_{n-1} e_{n-2} t_{n-2} \cdots t_1 e_0 t'_1 e_2 t_2 t'_3 \cdots t_{n-1} t'_n e_n$$

where t'_j is an appropriately chosen weak inverse of t_j and e_j is in the part of $\langle E(I) \rangle$ of I corresponding to the TCA block B_j. In fact, since for our particular I, TCA is the second power of the relation of being attached, each e_j can be taken to be a product of at most two idempotents.[3] The only multiplication involved which changes the group coordinate is the first multiplication by t_n, which places g in the group coordinate. The reader is invited to fill in the details; we recommend drawing many egg-box pictures.

We now aim to prove that $M_n \setminus t_j \in \mathbf{A} \vee \langle G \rangle$. Notice that $M_n \setminus t_j$ is a group mapping monoid with distinguished 0-minimal ideal I. Also $RLM_J(M_n \setminus t_j)$ is aperiodic being a submonoid of $RLM_J(M_n)$; in fact, $RLM_J(M_n \setminus t_j) = (M_n \setminus t_j)/\mathscr{H}$.

We need a lemma.

Lemma 5.3. *Suppose $\imath$ given by $(a,g,b) \mapsto (a, g_a g g_b, b)$ is a renormalization of a regular $\mathscr{J}$-class of a semigroup S and let $s \in S$ have wreath product action given by $(f_s, \cdot s)$ in the original coordinate system. Then the action in the new coordinate system is given by $(f'_s, \cdot s)$ where $bf'_s = g_b^{-1}(bf_s)g_{bs}$.*

Proof.

$$((a,g,b)s)\imath = (a, g(bf_s), bs)\imath = (a, g_a g(bf_s)g_{bs}, bs),$$

$$((a,g,b)s)\imath = (a, g_a g g_b, b)\imath s = (a, g_a g g_b(bf'_s), bs)$$

so we obtain

$$(bf_s)g_{bs} = g_b(bf'_s)$$

from which the lemma is easily established. □

Proposition 5.4. $M_n \setminus t_j \in \mathbf{A} \vee \langle G \rangle$.

Proof. There are two cases. First suppose $j = n$. Then it is clear from the construction of M_n and Lemma 3.4 that the hypotheses of Proposition 3.3 hold and we are done.

Suppose now that $j < n$. We think of Fig. 1 as a chain that we now break at t_j; see Fig. 2, where we take $j = 3$ for convenience.

We now perform a renormalization $(a,g,b) \mapsto (a, g_a g g_b, b)$ defined as follows. Set $g_b = g_i^{-1}$ if b belongs to a TCA block in the top row of Fig. 2 to the right of the

[3] K. Auinger has shown (private communication) one can choose $e_j \in E(I)$ for $j \neq 0$.

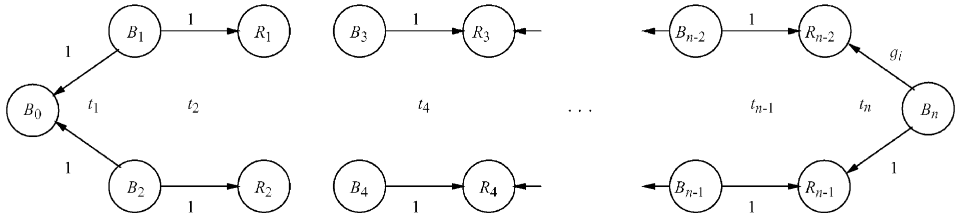

Fig. 2. Removal of t_j.

break, and $g_b = 1$, otherwise. Set $g_a = g_b^{-1}$ if a is attached to $b \in B$ (this is independent of the choice of b since the g_b depend only on the TCA block of b).

Let C' be the new matrix, cf. (3). Then, if a and b are attached,

$$bC'a = g_b^{-1}(bCa)g_a^{-1} = g_b^{-1}g_b = 1. \tag{13}$$

For $s \in S$, let us use $f_s' : B \to G$ for the wreath product action function with respect to the new coordinate system. By Lemma 3.4 and (13), for $s = (a_0, g_0, b_0) \in J$, $Bf_s' \subseteq \{g_0\}$, where we now use the new coordinate system for elements of J. Thus to show that Proposition 3.3 applies, it suffices to show that the t_k act as the identity on the group coordinate. A quick glance at Fig. 2 shows that if $s = t_k$ with $k < n$ and bs is defined, then $g_b = g_{bs}$ (it is here that we take advantage of the break). Moreover, since $bf_s = 1$, Lemma 5.3 tells us

$$bf_s' = g_b^{-1}(bf_s)g_{bs} = g_b^{-1}g_b = 1.$$

For $s = t_n$, we have (again dropping superscripts)

$$U_n f_s' = g_{U_n}^{-1}(U_n f_s)g_{R_{n-2}} = 1 g_i g_i^{-1} = 1,$$

$$V_n f_s' = g_{V_n}^{-1}(V_n f_s)g_{R_{n-1}} = 1.$$

We thus have $Bf_s \subseteq \{1\}$ for all $s \notin I$. Applying Proposition 3.3 gives the result. $\square$

We have completed the proof of Proposition 2.4, establishing all the results of Section 2.

Notice that $RLM_J(M_n) \setminus 1$ is an ideal extension of an aperiodic 0-simple semigroup with null quotient and so in the above proposition and all its corollary results (see Section 2), we may replace $\mathbf{A}$ with $\mathbf{ACS}^0 \, ⓜ \, \mathbf{N}_2$, as mentioned earlier.

6. Some local pseudovarieties

Contrary to the previous results, we show that pseudovarieties of the form $\mathbf{G} * \overline{\mathbf{H}}$ are always local if $\mathbf{H}$ is an extension-closed pseudovariety of groups. To do this we use the following important result of Karnofsky and Rhodes [11]. Recall [15] that if M is a monoid, then M^{RLM} is the direct sum of the right letter mapping representations over all regular $\mathscr{J}$-classes of M. The congruence $\equiv$ associated to M^{RLM} is the largest congruence with the property that $x \equiv y$ and x, y regular imply $x \mathscr{L} y$.

Theorem 6.1 (Karnofsky/Rhodes). *Let* $\mathbf{V}$ *be a pseudovariety of monoids closed under semidirect product. Then* $M \in \mathbf{G} * \mathbf{V}$ *if and only if* $M^{RLM} \in \mathbf{V}$.

The case of interest to Karnofsky and Rhodes was $\mathbf{V} = \mathbf{A}$. If $\mathbf{H}$ is a pseudovariety of groups closed under extension (or, equivalently, under semidirect product), then $\overline{\mathbf{H}}$ is also closed under semidirect product and so Theorem 6.1 applies.

Let C be a category. Recall that the consolidation C_{cd} of C is the semigroup obtained by adding a 0 to C and declaring all undefined products in C equal to 0. We denote by C_{cd}^1 the result of adjoining an identity to C_{cd} if it didn't already have one. Since $\overline{\mathbf{H}}$ contains the six element Brandt monoid B_2^1, the results of [27] show that $C \in \mathbf{g}(\mathbf{G} * \overline{\mathbf{H}})$ if and only if $C_{cd}^1 \in \mathbf{G} * \overline{\mathbf{H}}$.

Let c be an object in C and let C_c be the local monoid at c (i.e. $\mathrm{Hom}(c,c)$). We view C_c as a subsemigroup of C_{cd}^1 in the obvious way.

Lemma 6.2. *Let* $m,n \in C_c$. *Then* $m \mathscr{L} n$ *in* C_c *if and only if* $m \mathscr{L} n$ *in* C_{cd}^1. *A dual result holds for* $\mathscr{R}$.

Proof. Clearly $\mathscr{L}$-equivalence in C_c implies $\mathscr{L}$-equivalence in C_{cd}^1. For the converse, suppose $xm = n$ with $x \in C_{cd}^1$. If $m = n$, we are done. If $m \neq n$, then $x \neq 0, 1$—that is, $x \in C$. Then x must start at the initial object of n and end at the initial object of m. In other words, $x \in C_c$. Interchanging the roles of m and n, we obtain the desired result. $\square$

Lemma 6.3. *Suppose* $m,n \in C_c$. *Then* $mn \mathscr{J} m$ *in* C_{cd}^1 *if and only if* $mn \mathscr{J} m$ *in* C_c.

Proof. By stability of finite semigroups, $mn \mathscr{J} m$ if and only if $mn \mathscr{R} m$. The result now follows from Lemma 6.2. $\square$

Lemma 6.4. *Let* $\equiv$ *be the congruence on* C_c *associated to* C_c^{RLM} *and let* $\sim$ *be the congruence on* C_c *associated to the composition of the embedding* $C_c \hookrightarrow C_{cd}^1$ *and the quotient* $C_{cd}^1 \twoheadrightarrow (C_{cd}^1)^{RLM}$. *Then* $\equiv \, = \, \sim$.

Proof. Let $m,n \in C_c$. First note that $m \equiv n$ if and only if for all idempotents $e \in C_c$,

$$em \mathscr{J} e \iff en \mathscr{J} e \text{ in } C_c, \text{ and in this case } em \mathscr{L} en \text{ in } C_c. \tag{14}$$

On the other hand, $m \sim n$ if and only if for all idempotents $e \in C_{cd}^1$,

$$em \mathscr{J} e \iff en \mathscr{J} e \text{ in } C_{cd}^1, \text{ and in this case } em \mathscr{L} en \text{ in } C_{cd}^1. \tag{15}$$

First note that if $e \in C_{cd}$ is an idempotent, then either $e \in C_c$ or $em = 0 = en$. If $1 \notin C_{cd}$, then neither $1m$ nor $1n$ is $\mathscr{J}$-equivalent to 1. Thus when computing $\sim$, we only need to consider idempotents $e \in C_c$ in (15). But for $e \in C_c$, Lemmas 6.2 and 6.3 imply that (14) and (15) reduce to the same thing. $\square$

Theorem 6.5. *Let* $\mathbf{H}$ *be a pseudovariety of groups closed under semidirect product. Then* $\mathbf{G} * \overline{\mathbf{H}}$ *is local. In particular*, $\mathbf{G} * \mathbf{A}$ *is local.*

Proof. We show that if C is locally in $\mathbf{G} * \overline{\mathbf{H}}$, then $C^1_{cd} \in \mathbf{G} * \overline{\mathbf{H}}$, from which the result follows. By Theorem 6.1 it suffices to show $(C^1_{cd})^{RLM} \in \overline{\mathbf{H}}$.

Suppose $H \subseteq (C^1_{cd})^{RLM}$ is a non-trivial subgroup. Then standard techniques [15,8] imply that there is a (non-trivial) subgroup $G \subseteq C^1_{cd}$ mapping onto H in $(C^1_{cd})^{RLM}$. By the very construction of C^1_{cd}, G must be contained in C_c for some object c of C. Using both the notation and the result of Lemma 6.4, we obtain

$$H = G/\sim\, = G/\equiv .$$

Since $C_c \in \mathbf{G} * \overline{\mathbf{H}}$ by assumption, we have

$$C_c/\sim\, = C_c/\equiv = C_c^{RLM} \in \overline{\mathbf{H}}.$$

We conclude $H \in \mathbf{H}$. Since H was arbitrary, $(C^1_{cd})^{RLM} \in \overline{\mathbf{H}}$ and so $C^1_{cd} \in \mathbf{G} * \overline{\mathbf{H}}$, as desired. $\square$

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